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Exotic baryon multiplets at large number of colours

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Abstract

We generalize the usual octet, decuplet and exotic antidecuplet and higher baryon multiplets to any number of colours N_c. We show that the multiplets fall into a sequence of bands with $O(1/N_c)$ splittings inside the band and $O(1)$ splittings between the bands characterized by "exoticness", that is the number of extra quark-antiquark pairs needed to compose the multiplet. Each time one adds a pair the baryon mass is increased by the same constant which can be interpreted as a mass of a quark-antiquark pair. At the same time, we prove that masses of exotic rotational multiplets are reliably determined at large N_c from collective quantization of chiral solitons.

1 Introduction

The successful theoretical prediction of a relatively light and narrow exotic baryon Θ^+ [1] and its subsequent experimental observation [2, 3, 4, 5, 6] have stimulated much interest in the dynamics of baryons that cannot be made of three quarks.

Θ^+ has been predicted from a chiral soliton view on baryons, implying a semiclassical approximation. The approximation is justified at large N_c, although one puts $N_c = 3$ at the end of the calculations. At large N_c baryons *are* chiral solitons of some effective chiral action [7]. The question is whether this approach can give reliable description of the real $N_c = 3$ world. By reliable we mean that physical quantities like masses, widths, splittings etc., can be computed in the large-N_c limit, with known or at least controllable $1/N_c$ corrections. Much of this work has been done before for non-exotic baryons; our aim is to extend it to exotic baryons that cannot be composed of N_c quarks.

In order to understand the scaling of baryon properties with N_c one has to construct explicitly $SU(3)$ flavour multiplets (or representations) that are arbitrary-N_c prototypes of the lightest baryon multiplets – the octet with spin one half $\left(\mathbf{8}, \frac{1}{2}\right)$, the decuplet with spin three halves $\left(\mathbf{10}, \frac{3}{2}\right)$, the antidecuplet with spin one half $\left(\overline{\mathbf{10}}, \frac{1}{2}\right)$, etc. We generalize the previous study of this subject by Dulinski and Praszalowicz [8] and Cohen[9]. We classify the multiplets at arbitrary N_c by "exoticness" – the minimal number of extra quark-antiquark pairs one needs to add to the usual N_c quarks to build the multiplet. All multiplets that we shall be discussing appear as rotational excitations of a chiral soliton. We compute their energies and observe that the spectrum is equidistant in exoticness: each time one adds a pair it costs, at large N_c, a fixed energy independent of N_c. Being quite natural from the constituent quark point of view, this result is somewhat unusual for a rotational spectrum. Moreover, very recently Cohen [9] has expressed doubt whether the collective-quantization description of a particular exotic multiplet $\left(\overline{\mathbf{10}}, \frac{1}{2}\right)$ (to which the newly discovered Θ^+ presumably

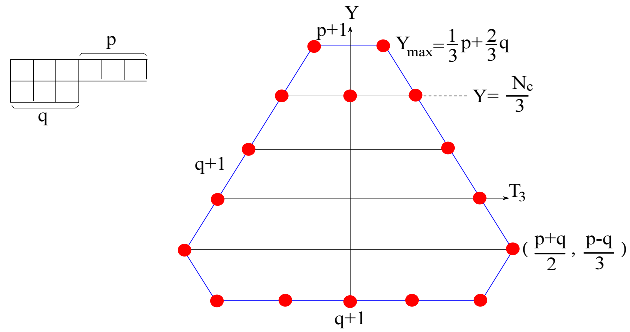

Figure 1: A weight diagram of a generic $SU(3)$ representation in the (T_3, Y) plane.

belongs) makes sense. In the second part of the paper we show that this doubt is ungrounded. Furthermore, we prove that the masses of exotic multiplets, obtained from the rotational spectrum, get only small $O(1/N_c)$ corrections from mixing with other, non-rotational degrees of freedom.

2 Generalization of baryon multiplets to arbitrary N_c

A generic $SU(3)$ multiplet (or irreducible representation) is uniquely determined by two non-negative integers (p, q) having the meaning of upper (lower) components of the irreducible $SU(3)$ tensor $T^{\{f_1 \dots f_p\}}_{\{g_1 \dots g_q\}}$ symmetrized both in upper and lower indices and with a contraction with any $\delta^{g_n}_{f_m}$ being zero. Schematically, q is the number of boxes in the lower line of the Young tableau depicting an $SU(3)$ representation and p is the number of extra boxes in its upper line. The dimension of a representation (or the number of particles in the multiplet) is

$$\mathrm{Dim}(p,q) = (p+1)(q+1)\left(1 + \frac{p+q}{2}\right) \tag{1}$$

and the eigenvalue of the quadratic Casimir operator is given by

$$C_2(p,q) = \frac{1}{3}\left[p^2 + q^2 + pq + 3(p+q)\right]. \tag{2}$$

The highest weight of a (p, q) representation (*i.e.* the state that generates all other states in the multiplet by applying 'eigenvalue-lowering' operators) is given by

$$\mathbf{e}_{\mathrm{H}}(p,q) = \left(T_3 = \frac{p+q}{2},\, Y = \frac{p-q}{3}\right) \tag{3}$$

where T_3 is the third projection of the isospin and Y is the hypercharge. On the weight (T_3, Y) diagram, a generic $SU(3)$ representation is depicted by a hexagon, whose upper (horizontal) side contains $p+1$ 'dots' or particles, the adjacent sides contain $q+1$ particles, with alternating $p+1$ and $q+1$ particles in the rest sides, the corners included – see Fig. 1. If either p or q is zero, the hexagon reduces to a triangle. Particles on the upper (horizontal) side of the hexagon have hypercharge

$$Y_{\max} = \frac{1}{3}p + \frac{2}{3}q \tag{4}$$

being the maximal possible hypercharge of a multiplet with given (p, q).

The quantization of the chiral soliton rotation in flavour and ordinary spaces proceeds as follows [10, 11, 12, 13, 14]. The lagrangian of the $SU(3)$ rotations is

$$\mathcal{L}_{\rm rot} = \frac{I_1}{2}\left(\Omega_1^2+\Omega_2^2+\Omega_3^2\right) + \frac{I_2}{2}\left(\Omega_4^2+\Omega_5^2+\Omega_6^2+\Omega_7^2\right) - \frac{N_c}{2\sqrt{3}}\Omega_8 \tag{5}$$

where Ω_A are angular velocities of the soliton and $I_{1,2}$ are the two soliton moments of inertia, depending on its concrete dynamical realization. Rotation along the 8th axis in flavour space commutes with the 'upper-left-corner' Ansatz for the soliton field, therefore there is no quadratic term in Ω_8. However there is a Wess–Zumino–Witten term linear in Ω_8. The canonical quantization leads to the hamiltonian

$$\mathcal{H}_{\rm rot} = \frac{J_1^2+J_2^2+J_3^2}{2I_1} + \frac{J_4^2+J_5^2+J_6^2+J_7^2}{2I_2}, \tag{6}$$

where the angular momenta satisfy the $SU(3)$ commutation relations. Eq. (6) must be supplemented by the quantization condition $J_8 = -N_c/2\sqrt{3} = -Y'\sqrt{3}/2$ following from the Wess–Zumino–Witten term. Given that

$$\sum_{A=1}^{3} J_A^2 = J(J+1), \qquad \sum_{A=1}^{8} J_A^2 = C_2(p,q), \qquad J_8^2 = \frac{N_c^2}{12}, \tag{7}$$

one gets the rotational energy of baryons with given spin J and belonging to representation (p,q):

$$\mathcal{E}_{\rm rot}(p,q,J) = \frac{C_2(p,q) - J(J+1) - \frac{N_c^2}{12}}{2I_2} + \frac{J(J+1)}{2I_1}. \tag{8}$$

Only those multiplets are realized as rotational excitations that have members with hypercharge $Y = \frac{N_c}{3}$; if the number of particles with this hypercharge is n the spin J of the multiplet is such that $2J+1 = n$. It is easily seen that the number of particles with a given Y is $\frac{4}{3}p + \frac{2}{3}q + 1 - Y$ and hence the spin of the allowed multiplet is

$$J = \frac{1}{6}(4p + 2q - N_c). \tag{9}$$

A common mass $\mathcal{M}_0$ must be added to eq. (8) to get the mass of a particular multiplet. Throughout this paper we are disregarding the splittings inside multiplets as due to non-zero current strange quark mass.

The condition that a horizontal line $Y = \frac{N_c}{3}$ must be inside the weight diagram for the allowed multiplet leads to the requirement

$$\frac{N_c}{3} \le Y_{\rm max} \qquad \text{or} \qquad p + 2q \ge N_c \tag{10}$$

showing that at large N_c multiplets must have a high dimension.

We introduce a non-negative number which we name "exoticness" E of a multiplet defined as

$$Y_{\rm max} \;=\; \frac{1}{3}p + \frac{2}{3}q \;\equiv\; \frac{N_c}{3} + E, \qquad E \ge 0. \tag{11}$$

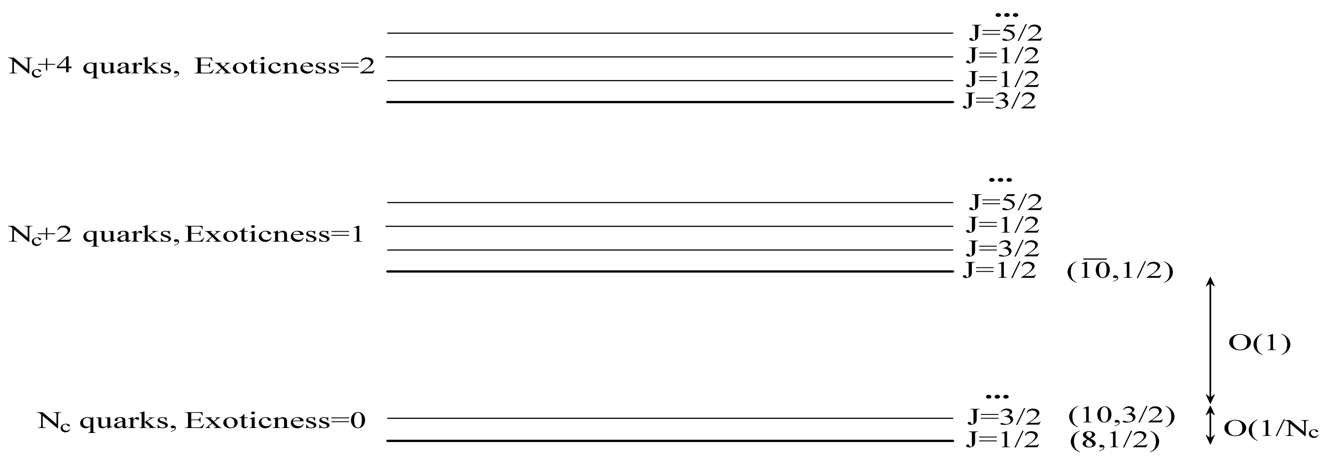

Figure 2: Rotational excitations form a sequence of bands.

Combining eqs.(9,11) we express (p, q) through (J, E):

$$\begin{aligned} p &= 2J - E, \\ q &= \frac{1}{2}N_c + 2E - J. \end{aligned} \tag{12}$$

The total number of boxes in Young tableau is $2q + p = N_c + 3E$. Since we are dealing with unit baryon number states, the number of quarks in the multiplets we discuss is N_c, *plus* some number of quark-antiquark pairs. In the Young tableau, quarks are presented by single boxes and antiquarks by double boxes. It explains the name "exoticness": E gives the minimal number of additional quark-antiquark pairs one needs to add on top of the usual N_c quarks to compose a multiplet.

Putting (p, q) from eq. (12) into eq. (8) we obtain the rotational energy of a soliton as function of the spin and exoticness of the multiplet:

$$\mathcal{E}_{\rm rot}(J, E) = \frac{E^2 + E(\frac{N_c}{2} + 1 - J) + \frac{N_c}{2}}{2I_2} + \frac{J(J+1)}{2I_1}. \tag{13}$$

We see that for given $J \leq \frac{N_c}{2} + 1$ the multiplet mass is a monotonically growing function of E: the minimal-mass multiplet has $E = 0$. Masses of multiplets with increasing exoticness are:

$$\mathcal{M}_{\rm E=0}(J) = \mathcal{M}_0' + \frac{J(J+1)}{2I_1}, \qquad \text{where} \quad \mathcal{M}_0' \equiv \mathcal{M}_0 + \frac{N_c}{4I_2}, \tag{14}$$

$$\mathcal{M}_{\rm E=1}(J) = \mathcal{M}_0' + \frac{J(J+1)}{2I_1} + 1 \cdot \frac{\frac{N_c}{2} + 2 - J}{2I_2}, \tag{15}$$

$$\mathcal{M}_{\rm E=2}(J) = \mathcal{M}_0' + \frac{J(J+1)}{2I_1} + 2 \cdot \frac{\frac{N_c}{2} + 2 - J}{2I_2} + \frac{1}{I_2}, \tag{16}$$

$$\mathcal{M}_{\rm E=3}(J) = \mathcal{M}_0' + \frac{J(J+1)}{2I_1} + 3 \cdot \frac{\frac{N_c}{2} + 2 - J}{2I_2} + \frac{3}{I_2}, \qquad \text{etc.} \tag{17}$$

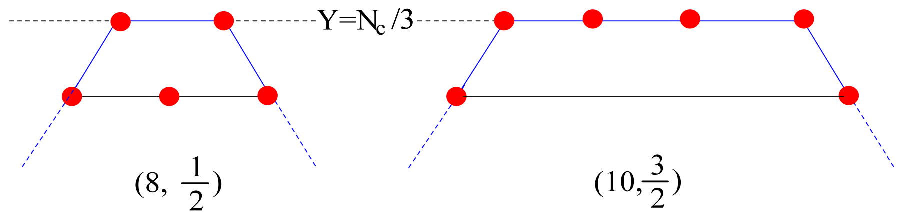

Figure 3: Non-exotic ($E=0$) multiplets that can be composed of N_c quarks.

At this point it should be recalled that both moments of inertia $I_{1,2} = O(N_c)$, as is $\mathcal{M}_0$. We see from eqs.(14)-(17) that multiplets fall into a sequence (labelled by exoticness) of rotational bands with small $O(1/N_c)$ splittings inside the bands; the separation between bands with different E is $O(1)$. The corresponding masses are schematically shown in Fig. 2.

The lowest band is non-exotic ($E=0$); the multiplets are determined by $(p,q) = \left(2J, \frac{N_c}{2} - J\right)$, and their dimension is $Dim = (2J+1)(N_c+2-2J)(N_c+4+2J)/8$ which in the particular (but interesting) case of $N_c = 3$ becomes 8 for spin one half and 10 for spin 3/2. These are the correct lowest multiplets in real world, and the above multiplets are their generalization to arbitrary values of N_c. To make baryons fermions one needs to consider only odd N_c.

Recalling that u, d, s quarks' hypercharges are 1/3, 1/3 and -2/3, respectively, one observes that all baryons of the non-exotic $E=0$ band can be made of N_c quarks. The upper side of their weight diagrams (see Fig. 3) is composed of u, d quarks only; in the lower lines one consequently replaces u, d quarks by the s one. This is how the real-world $\left(\mathbf{8}, \frac{1}{2}\right)$ and $\left(\mathbf{10}, \frac{3}{2}\right)$ multiplets are arranged and this property is preserved in their higher-N_c generalizations. The construction coincides with that of ref. [8]. At high N_c there are further multiplets with spin 5/2 and so on. The maximal possible spin at given N_c is $J_{\max} = \frac{N_c}{2}$: if one attempts higher spin, q becomes negative.

The rotational bands for $E=1$ multiplets are shown in Fig. 4. The upper side of the weight diagram is exactly one unit higher than the line $Y = \frac{N_c}{3}$ which is non-exotic, in the sense that its quantum numbers can be, in principle, achieved from exactly N_c quarks. However, particles corresponding to the upper side of the weight diagram cannot be composed of N_c quarks but require at least one additional $\bar{s}$ quark and hence *one additional quark-antiquark pair* on top of N_c quarks.

The multiplet shown in Fig. 4, left, has only one particle with $Y = \frac{N_c}{3} + 1$. It is an isosinglet with spin $J = \frac{1}{2}$, and in the quark language is built of $(N_c+1)/2$ ud pairs and one $\bar{s}$ quark. It is the generalization of the Θ^+ baryon to arbitrary odd N_c. As seen from eqs.(1,12), the multiplet to which

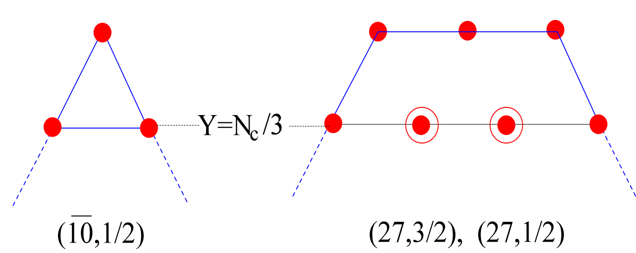

Figure 4: Exotic ($E=1$) multiplets that can be composed of N_c quarks and one extra pair.

the "Θ^+" belongs is characterized by $(p,q) = (0,(N_c+3)/2)$, its dimension is $(N_c+5)(N_c+7)/8$ becoming the $\left(\overline{\mathbf{10}},\frac{1}{2}\right)$ at $N_c = 3$. Its splitting with the N_c generalization of the non-exotic $\left(\mathbf{8},\frac{1}{2}\right)$ multiplet follows from eq. (15):

$$\mathcal{M}_{\overline{10},\frac{1}{2}} - \mathcal{M}_{8,\frac{1}{2}} = \frac{N_c+3}{4I_2} \tag{18}$$

coinciding with the recent finding of ref. [9]. Here and in what follows we denote baryon multiplets by their dimension at $N_c{=}3$ although at $N_c{>}3$ their dimension is higher, as given by eq. (1).

The second rotational state of the $E{=}1$ sequence has $J = \frac{3}{2}$; it has $(p,q) = (2,(N_c+1)/2)$ and dimension $3(N_c+3)(N_c+9)/8$ reducing to the mutliplet $\left(\mathbf{27},\frac{3}{2}\right)$ at $N_c = 3$, see Fig. 4, right. In fact there are two physically distinct multiplets there. Indeed, the weights in the middle of the second line from top on the weight diagram (with $Y = \frac{N_c}{3}$) are twice degenerate, corresponding to spin 3/2 and 1/2. Therefore, there is another $3(N_c+3)(N_c+9)/8$-plet with unit exoticness, but with spin 1/2. At $N_c{=}3$ it reduces to $\left(\mathbf{27},\frac{1}{2}\right)$. The splittings with non-exotic multiplets are

$$\mathcal{M}_{27,\frac{3}{2}} - \mathcal{M}_{10,\frac{3}{2}} = \frac{N_c+1}{4I_2}, \tag{19}$$

$$\mathcal{M}_{27,\frac{1}{2}} - \mathcal{M}_{8,\frac{1}{2}} = \frac{N_c+7}{4I_2}. \tag{20}$$

The $E{=}1$ band continues to the maximal spin $J_{\max} = (N_c+4)/2$ where q becomes zero.

These higher multiplets in the rotational spectrum of the $SU(3)$ soliton at $N_c = 3$ has been known to the skyrmion community from the 1980's. After the discovery of Θ^+ there has been a renewed interest in them [15, 16].

The $E = 2$ rotational band starts from two states with spin 3/2 and 1/2 both belonging to the $SU(3)$ representation $(p,q,Dim) = (1,(N_c+5)/2,(N_c+7)(N_c+11)/4)$. It reduces to the $\overline{\mathbf{35}}$ multiplet at $N_c{=}3$. Their splittings with non-exotic multiplets are

$$\mathcal{M}_{\overline{35},\frac{3}{2}} - \mathcal{M}_{10,\frac{3}{2}} = \frac{N_c+3}{2I_2}, \tag{21}$$

$$\mathcal{M}_{\overline{35},\frac{1}{2}} - \mathcal{M}_{8,\frac{1}{2}} = \frac{N_c+6}{2I_2}. \tag{22}$$

The maximal spin of the $E{=}2$ rotational band is $J_{\max} = (N_c+8)/2$.

The upper side in the weight diagram (Fig. 5) for the $E = 2$ sequence has hypercharge $Y_{\max} = \frac{N_c}{3} + 2$. Therefore, one needs *two* $\bar{s}$ quarks to get that hypercharge and hence the multiplets can be minimally constructed of N_c quarks plus *two additional quark-antiquark pairs*. This explains the name "exoticness" E: it gives the minimal number of additional quark-antiquark pairs needed to construct a multiplet, on top of the usual N_c quarks. It is also seen from counting the number of boxes in the Young tableau.

Disregarding the rotation along the 1,2,3 axes (for example taking only the lowest J state from each band) we observe from eq. (**??**) that at large N_c the spectrum is equidistant in exoticness,

$$\mathcal{E}_{\rm rot}(E) = \frac{N_c(E+1)}{4I_2}, \tag{23}$$

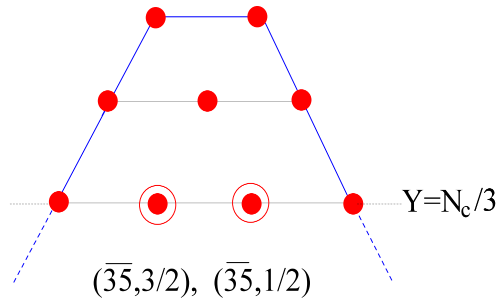

Figure 5: Two lowest exotic ($E=2$) multiplets that can be composed of N_c quarks and two extra pairs.

with the spacing $\frac{N_c}{4I_2}=O(1)$. It means that each time we add a quark-antiquark pair it costs at large N_c the same energy

$$\text{energy of a pair} = \frac{N_c}{4I_2} = O(N_c^0). \tag{24}$$

Naively one may think that this quantity should be approximately twice the constituent quark mass. Actually, it can be much less than that. For example, in the Chiral Quark Soliton Model [17, 18] an inspection of I_2 given there shows that the pair energy is strictly less than $2M$; in fact $1/I_2$ tends to zero in the non-relativistic limit of the model. In physical terms, the energy cost of adding a pair can be small if the pair is added in the form of a Goldstone boson.

3 Why collective quantization is valid for exotic multiplets

Eq. (23) is interesting as it says that the rotational energy corresponding to exotic baryons is of the same order as the expected vibrational or radial excitation energies. Therefore one can suspect that rotational excitations might mix up with those other ones, and no reliable predictions for exotic baryons, based on the collective quantization of baryon rotation, can be made. This doubt has been formulated by Cohen [9] who writes: "The key point is that a collective description is valid only for motion which is slow compared to vibrational modes which are of order N_c^0 ... The characteristic time scale of quantized collective motion is given by the typical quantum mechanical result $\tau \sim (\Delta\mathcal{E})^{-1}$, where $\Delta\mathcal{E}$ is the splitting between two neighboring collective levels." Since the splittings are of the same order for vibrations and rotations he concludes that the rotational description of exotics is not valid [1].

We shall demonstrate that the characteristic time for rotation is in fact much larger than Cohen's estimate. Next we calculate the vibration-rotation mixing and prove that it is small at large N_c.

One can find directly the rotation time from angular velocities. Unusual as it might be, the estimate of the time scale for rotation from level splitting turns out to be wrong in this particular case: actually the rotation is much slower.

From eq. (13) one immediatelly gets the sum of the squares of angular momenta for a particular

[1] In ref. [9] only the splitting of the "antidecuplet" has been considered but not the general case.

(J, E) multiplet:

$$\sum_{A=1}^{3} J_A^2 \;=\; J(J+1), \tag{25}$$

$$\sum_{A=4}^{7} J_A^2 \;=\; E^2 + E(\frac{N_c}{2} + 1 - J) + \frac{N_c}{2} \;\overset{N_c\to\infty}{\longrightarrow}\; \frac{N_c}{2}(E+1), \tag{26}$$

meaning that $J_{4\text{–}7} = O(\sqrt{N_c})$ even for zero exoticness. This should be contrasted to the $SU(2)$ angular momenta: $J_{1\text{–}3} = O(1)$. For this reason the splittings inside a band with given E are $O(1/N_c)$ whereas the splitting between the bands with growing exoticness is $O(1)$.

Let us estimate the angular velocities. As operators, $\hat{\Omega}_{1\text{–}3} = \hat{J}_{1\text{–}3}/I_1$ and $\hat{\Omega}_{4\text{–}7} = \hat{J}_{4\text{–}7}/I_2$. Since positive-definite sums of angular momenta's squares are fixed by eq. (7) one obtains

$$\sqrt{<\Omega_{1\text{–}3}^2>} \;\leq\; \frac{\sqrt{J_1^2+J_2^2+J_3^2}}{I_1} = O\left(\frac{J}{N_c}\right), \tag{27}$$

$$\sqrt{<\Omega_{4\text{–}7}^2>} \;\leq\; \frac{\sqrt{J_4^2+J_5^2+J_6^2+J_7^2}}{I_2} = O\left(\frac{\sqrt{E+1}}{\sqrt{N_c}}\right). \tag{28}$$

We see that angular velocities $\Omega_{4\text{–}7} \gg \Omega_{1\text{–}3}$ at large N_c even for non-exotic multiplets ($E=0$). From this point of view exotic and non-exotic multiplets are on equal footing. It is related to that all multiplets have high dimensions at large N_c. However, $\Omega_{4\text{–}7}$ *are small for any exoticness* $E \ll N_c$, and the soliton rotation for such multiplets is slow.

Normally, eq. (23) would imply from the semiclassical Bohr's correspondence principle that the characteristic time scale for rotation is $\tau \sim (\Delta\mathcal{E})^{-1} = O(1)$, as assumed in ref. [9]. From the explicit calculation above we see, however, that the characteristic time scale for rotation is instead $\tau \sim (\Omega_{4\text{–}7})^{-1} = O(\sqrt{N_c})$. This is a rather rare occasion when the estimate of the time scale from the semiclassical Bohr's correspondence principle is incorrect. The reason is that in this particular case the rotation is not semiclassical because of the quantization condition $J_8 = -N_c/2\sqrt{3}$. It arises from the Wess–Zumino–Witten term in eq. (5) which is a full derivative and, classically, should be thrown out.

The characteristic time scale for rotation leading to exotic multiplets is thus large, $\tau = O(\sqrt{N_c})$. In particular, rotation is much slower compared to the vibrational or radial modes whose excitation energies are $O(1)$ and hence their time scale is $O(1)$. Therefore, it is fully legitimate to treat the rotation leading to exotic multiplets as collective excitations of a chiral soliton, at least as long as $E \ll N_c$.

To make this statement quantitative, we derive the shift in the exotic baryons' rotation energy as due to mixing with non-rotational degrees of freedom. We show that this shift is $O(1/N_c)$ and tends to zero at large N_c. It is a direct proof that exotic multiplets can be treated as collective excitations. We first present a general derivation of the mixing and then give a more transparent quantum mechanical example with only one non-rotational degree of freedom but preserving all properties of the case at hand. To purify the argument we imply here an idealized case of $N_c \to \infty$ and neglect non-zero m_s.

General chiral soliton derivation

At large N_c baryons are chiral solitons [7] corresponding to a local minimum of some effective chiral action $S_{\rm eff}[\pi(x)]$. Its specific form is irrelevant to the argument: the important thing is that it is proportional to N_c. Its local minimum $\pi_{\rm class}(x)$ gives the soliton profile, and the moments of inertia $I_{1,2}$ are computed at this minimum. Hence $\mathcal{M}_0$, $I_{1,2}$ are all proportional to N_c.

Vibrational modes of a baryon are encoded in quantum fluctuations about the classical minimum. One expands the static energy part of the effective chiral action about the minimum:

$$\mathcal{E}_{\rm eff}[\pi_{\rm class}+\delta\pi]=\mathcal{M}_0+\frac{1}{2}\delta\pi\, W[\pi_{\rm class}]\,\delta\pi+\dots \tag{29}$$

where W is some operator in a given external field $\pi_{\rm class}$ and is usually referred to as the quadratic form. It is of the order of N_c (since the full $S_{\rm eff}$ is), hence quantum fluctuations scale as $\delta\pi(x)=O(1/\sqrt{N_c})$. The spectrum and eigenfunctions of W are N_c-independent. W has zero modes related to symmetry, in this case translations and rotations. The quantization of rotations (which are large fluctuations as they occur in flat zero-mode directions) leads to the rotational spectrum discussed in the previous section. The vibrational modes are orthogonal to those zero modes. One can expand a general fluctuation in eigenfunctions of the quadratic form:

$$W\psi_n(x) = \kappa_n\psi_n(x), \qquad \kappa_n>0, \tag{30}$$

$$\delta\pi(x) = \sum_n c_n\psi_n(x). \tag{31}$$

Assuming eigenfunctions are (ortho) normalized to unity, the Fourier coefficients are $c_n=O(1/\sqrt{N_c})$ and the eigenvalues are $\kappa_n=O(N_c)$ since W is proportional to N_c. c_n's can be considered as normal coordinates for vibrations. In the harmonic approximation their hamiltonian is

$$\mathcal{H}_{\rm vibr}=\sum_n\left(-\frac{1}{2\mu_n}\frac{\partial^2}{\partial c_n^2}+\frac{\kappa_n}{2}c_n^2\right) \tag{32}$$

with $\kappa_n,\mu_n=O(N_c)$ leading to vibration energies $\epsilon_{n,k_n}=\sqrt{\kappa_n/\mu_n}(k_n+\frac{1}{2})=O(1)$, as it should be. The ground-state ($k_n{=}0$) wave function is $\psi(c_n)=\exp(-\sqrt{\kappa_n\mu_n}c_n^2/2)$.

We now consider the influence of vibrations on the rotational spectrum $\mathcal{E}_{\rm rot}=\frac{N_c(E+1)}{4I_2[\pi]}$. In the leading (classical) order one substitutes the classical soliton field and gets the moment of inertia $I_{20}=I_2[\pi_{\rm class}]=O(N_c)$. Taking into account quantum fluctuations one expands

$$I_2[\pi_{\rm class}{+}\delta\pi]=I_{20}{+}\delta I_2{+}\delta^2 I_2{+}\dots=I_{20}{+}\sum\alpha_n c_n{+}\sum_{m,n}\beta_{mn}c_m c_n{+}\dots, \qquad \alpha_n,\beta_{mn}=O(N_c). \tag{33}$$

Consequently, the hamiltonian for rotation-vibration mixing is

$$\mathcal{H}_{\rm rot-vibr}=\frac{N_c(E+1)}{4I_2[\pi_{\rm class}+\delta\pi]}=\frac{N_c(E+1)}{4I_{20}}\left[1-\frac{\sum_n\alpha_n c_n}{I_{20}}+\frac{\sum_{m,n}(\alpha_m\alpha_n-\beta_{mn}I_{20})c_m c_n}{I_{20}^2}+\dots\right]. \tag{34}$$

One can now evaluate the corresponding rotation-vibration mixing energy by perturbation theory in quantum fluctuations c_n. The term linear in c_n is zero in the first order but in the second order perturbation theory it is non zero and one gets

$$\mathcal{E}^{(1)}_{\rm rot-vibr} \simeq \mathcal{E}^2_{\rm rot}\frac{<1|\sum\alpha_n c_n|0>^2}{\Delta\mathcal{E}_{\rm vibr}\, I_{20}^2} \simeq \mathcal{E}^2_{\rm rot}\sum_n \frac{1}{\kappa_n}\left(\frac{\alpha_n}{I_{20}}\right)^2 \simeq \mathcal{E}_{\rm rot}\,\frac{\mathcal{E}_{\rm rot}}{\Delta\mathcal{E}_{\rm vibr}}\left\langle\left(\frac{\delta I_2}{I_2}\right)^2\right\rangle \propto O\left(\frac{(E+1)^2}{N_c}\right) \tag{35}$$

which is a small $1/N_c$ correction to the main rotational energy, eq. (23). Only when exoticness is comparable to N_c the correction becomes of the same order as the main term.

Another contribution to the mixing arises from the last term in eq. (34). Here it is sufficient to use the first order perturbation theory, and one obtains

$$\mathcal{E}^{(2)}_{\rm rot-vibr} = \mathcal{E}_{\rm rot}\sum_n \frac{1}{\sqrt{\kappa_n\mu_n}}\frac{\alpha_n^2-\beta_{nn}I_{20}}{I_{20}^2} \simeq \mathcal{E}_{\rm rot}\left\langle\left(\frac{\delta I_2}{I_2}\right)^2 - \frac{\delta^2 I_2}{I_2}\right\rangle \propto O\left(\frac{E+1}{N_c}\right). \tag{36}$$

This term is a small correction to $\mathcal{E}_{\rm rot}$ (23) even at large exoticness. Thus we have proved that, despite the rotational energy for exotic baryons being $O(1)$, its mixing with vibrational degrees of freedom leads to a small $O(1/N_c)$ correction.

The change in the baryon form owing to rotation cannot be neglected only when the rotational energy reaches $\mathcal{E}_{\rm rot} = O(N_c)$ comparable to the static baryon mass $\mathcal{M}_0$. At this point everything goes wrong [19, 20]: the widths become comparable to the masses owing to strong pion radiation and the centrifugal forces deform the baryon such that the rotational energy has to be computed anew.

Charged particle in the field of a monopole

To illustrate the general derivation above, we consider a quantum-mechanical example preserving the equidistant rotational spectrum with the spacing of the same order as in the vibrational one. In fact we take the example of Guadagnini [10] which he used as analogy to derive the quantization of the $SU(3)$ skyrmion. This example is an ideal copy of the case at hand, except that there is only one non-rotational degree of freedom instead of an infinite number as in the real case. At the same time it is so simple that all calculations can be done explicitly and to any given accuracy.

Consider a charged particle on a sphere of radius R surrounding a magnetic monopole whose magnetic field is $\mathbf{B} = \mathbf{r}/r^3$. The lagrangian can be written in terms of the angular velocities of the particle $\Omega_{1,2,3}$ with a "Wess–Zumino–Witten" term linear in Ω_3 [10, 11]:

$$\mathcal{L}_{\rm rot} = \frac{I}{2}(\Omega_1^2+\Omega_2^2) + eg\,\Omega_3, \tag{37}$$

where eg is the product of the electric and magnetic charges; it must be an integer. The canonical quantization leads to the hamiltonian written in terms of orbital momentum operators

$$\mathcal{H}_{\rm rot} = \frac{1}{2I}(L_1^2+L_2^2) \tag{38}$$

supplemented with the quantization condition $L_3 = eg$, similar to what one requires in eq. (6). The hamiltonian has eigenvalues $\mathcal{E}_{\rm rot} = [L(L+1)-(eg)^2]/2I$. At large $N = eg$ (being a direct analog of large N_c) we introduce an analog of exoticness $E \equiv L - N$ and rewrite the rotational energy as

$$\mathcal{E}_{\rm rot} = \frac{E^2+E(2N+1)+N}{2I}. \tag{39}$$

This is fully similar to eq. (13): at large N but finite E the rotational spectrum is equidistant with the spacing $\Delta\mathcal{E} = N/I$, and we imply that the moment of inertia is $I = \mu R^2 = O(N)$, with $R = O(1)$ and particle mass $\mu = O(N)$.

To study the interplay between rotations and vibrations, we now allow the particle to deviate from the sphere of radius L putting it in a potential well $\kappa(|\mathbf{r}| - R)^2/2$. We arrive at the hamiltonian [10]

$$\mathcal{H} = -\frac{1}{2\mu}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right) + \frac{(2E+1)N}{2\mu r^2} + \frac{\kappa(r-R)^2}{2} \tag{40}$$

where we take $\kappa = O(N)$ to have vibrational excitations stable in N.

If one neglects the fluctuations $x \equiv r - R$ as compared to the average distance R, the rotation and vibration variables are completely separated and one obtains

$$\begin{aligned}
\mathcal{E}_{\rm rot}(E) &= \frac{(2E+1)N}{2I} = O(1), \qquad \text{moment of inertia } I = \mu R^2,\\
\mathcal{E}_{\rm vibr}(k) &= \sqrt{\frac{\kappa}{\mu}}\left(k + \frac{1}{2}\right) = O(1), \qquad \psi(x) = \exp\left(-\sqrt{\kappa\mu}\,\frac{x^2}{2}\right),\\
<x^2> &\simeq \frac{1}{\sqrt{\kappa\mu}} = O\left(\frac{1}{N}\right).
\end{aligned}$$

The interaction between rotations and vibrations comes through the centrifugal term in eq. (40). We expand $1/(R+x)^2 = 1/R^2(1 - 2x/R + 3x^2/R^2 + \ldots)$ and estimate it using the harmonic oscillator wave functions in x. The leading term is zero by parity, therefore we have to apply the second-order perturbation theory for the x term and the first-order perturbation theory for the x^2 term. The former gives

$$\mathcal{E}^{(1)}_{\rm rot-vibr} \simeq \mathcal{E}_{\rm rot}\frac{\mathcal{E}_{\rm rot}}{\Delta\mathcal{E}_{\rm vibr}}\frac{<1|x|0>^2}{R^2} \simeq \mathcal{E}^2_{\rm rot}\frac{1}{\kappa R^2} \propto \frac{(2E+1)^2}{N}. \tag{41}$$

The latter gives

$$\mathcal{E}^{(2)}_{\rm rot-vibr} \simeq \mathcal{E}_{\rm rot}\frac{<x^2>}{R^2} \simeq \mathcal{E}_{\rm rot}\frac{1}{R^2\sqrt{\kappa\mu}} \propto \frac{2E+1}{N}. \tag{42}$$

It is always a small correction. As to eq. (41), it becomes a sizable correction only when the 'exoticness' compares to N, like in the above general analysis.

In this example the rotational splittings are themselves $O(1)$, like the splittings between baryons with different exoticness. Nevertheless, the "baryon masses" are computed accurately from the rotational spectrum. Furthermore, one sees that the rotational level spacing is actually irrelevant to the shift of the rotational energy owing to vibrations. The only thing which counts is the *change of the moment of inertia*. This change is $\delta I/I = O(1/N_c)$ in the general case and in the example considered. The change $\delta I/I$ becomes of the order of unity in two cases: *i*) when the vibrational excitation is of the order of N_c so that $\delta x/R = O(1)$, *ii*) when rotational energy is of the order of N_c so that $\mathcal{E}_{\rm rot}/\mathcal{E}_{\rm vibr} = O(N_c)$. In such cases the deformation of the soliton is not small, and one cannot consider it as a rigid body.

Finally, let us comment on the recent suggestion [9] that Θ^+ could be extracted from a linear response theory describing meson-soliton scattering [21, 22]. Essentially the same idea is proposed in ref. [23] to find Θ^+ partner states, using the chiral-soliton formalism of ref. [24].

The spectrum of vibrational modes is defined by the quadratic form W. In a specific (Skyrme) model this spectrum has been studied in refs. [21, 22]. The spectrum is naturally N_c-independent, as are the ensuing meson-baryon scattering amplitudes. However it does not mean that *all* N_c-independent excitations will be seen as poles in the scattering amplitudes generated by the quadratic form. It is well known that poles related to rotational excitations are missed in the quadratic form: they correspond to fluctuations in flat zero-mode directions, and cannot be considered as small. All rotational states, independently of their energy, arise as poles in meson-baryon scattering amplitudes from Born graphs which are missing in the quadratic form but have been recovered in ref. [25] as a purely classical effect. In particular, assuming Θ^+ is a $O(1)$ rotational excitation, it will not occur in the small-oscillation spectrum, contrary to the statement of refs. [9, 23].

4 Summary

We have constructed the generalization of $\left(\mathbf{8},\frac{1}{2}\right)$, $\left(\mathbf{10},\frac{3}{2}\right)$, $\left(\overline{\mathbf{10}},\frac{1}{2}\right)$, $\left(\mathbf{27},\frac{3}{2}\right)$, $\left(\mathbf{27},\frac{1}{2}\right)$... multiplets to the case of arbitrary N_c. These multiplets are classified by “exoticness” – the number of extra quark-antiquark pairs needed to compose the multiplet. The splittings between masses of the multiplet with the same exoticness are $O(1/N_c)$ *i.e.* parametrically small at large N_c. The spectrum of multiplets with growing exoticness is equidistant with an $O(1)$ spacing. On the one hand this spacing can be interpreted as an energy for adding a quark-antiquark pair but on the other hand it is a rotational excitation. Despite that it becomes comparable to vibrational or radial excitations of baryons, both non-exotic and exotic bands are, at large N_c, reliably described as collective excitations of the ground-state baryons: the corrections die out as $1/N_c$. This conclusion is opposite to the recent claim of ref. [9]. That claim has been based on an assumption that the rotation corresponding to exotic baryons is fast but in fact it is slow, as we have explicitly shown. The collective quantization description fails only when the exoticness becomes comparable to N_c.

The newly discovered Θ^+ baryon belongs to the exoticness=1 multiplet $\left(\overline{\mathbf{10}},\frac{1}{2}\right)$. The larger N_c the more accurate would be its description as a rotational state of a chiral soliton.

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